Math 210C Lecture 24 Notes

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1 Representations and Maschke's Theorem

1.1 Representations

Let F be a field, G be a group, and V be a finite dimensional vector space. An $F[G]$-module structure on V is the same as a homomorphism $\rho : G\ to\, \mathrm{Aut}_F(V)$. IF $V = F^n$, then $\mathrm{Aut}_F(V) \cong \mathrm{GL}_n(F)$ by choosing the standard basis.

Definition 1.1. A **representation** of G over F (or an F**-representation** of G) is an F-vector space V together with a homomorphism $\rho : G|to\, \mathrm{Aut}_F(V)$.

We sometimes write ρ_V to denote the homomorphism associated to the representation V.

Example 1.1. If $G \leq \mathrm{GL}_n(F)$, then the inclusion homomorphism $\rho : G \to \mathrm{GL}_n(F)$ is a representation. Then $\mathrm{GL}_n \circlearrowright F^n$ by left multiplication, and this restricts to an action $G \circlearrowright F^n$.

Example 1.2. We have the permutation representation $S_n \to \mathrm{GL}_n(\mathbb{Z}) \to \mathrm{GL}_n(F)$, which sends a permutation to its associated permutation matrix.

Example 1.3. We have $\rho : \mathbb{R} \to \mathrm{GL}_2(\mathbb{R})$ given by

$$\rho(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Definition 1.2. The **trivial representation** is F with the trivial G-action: $\rho : G \to F^\times$ sends $g \mapsto 1$.

Definition 1.3. The **regular representation** is $F[G]$ as a left $F[G]$ module.

If G is finite, then the dimension of the representation is $|G|$.

Definition 1.4. Two representations are **isomorphic** (or **conjugate**) if their underlying $F[G]$-modules are isomorphic: $\rho_w(g) = \varphi \circ \rho_V(g) \circ \varphi^{-1}$

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\rho_V"] & W \arrow[d, "\rho_W"] \\ F^n \arrow[r] & F^n \end{tikzcd}$$

We can speak or representations being **irreducible** (simple), semisimple, or indecomposable.

Definition 1.5. A **subrepresentation** is an $F[G]$-submodule.

Example 1.4. 1-dimensional representations are irreducible.

Example 1.5. Let $D_p = \langle r, s \rangle$, where p is prime be a dihedral group. We have the representation $\rho : D_p \to \mathrm{GL}_2(\mathbb{F}_p)$ sending

$$\rho(s) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \rho(r) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

This is injective, so the representation is **faithful**.

The representation ρ is indecomposable but not irreudcible: $\mathbb{F}_p \cdot e_1$ is $\mathbb{F}_p[D_p]$-stable, but it does not have a complement. Any $v \notin \mathbb{F}_p \cdot e_1$ spans $\mathbb{F}_p^2$ as an $\mathbb{F}_p[D_p]$-module.

1.2 Maschke's theorem and decomposition into irreducible representations

Theorem 1.1 (Maschke)**.** *Let G be a finite group, let F be a field of* $\mathrm{char} \nmid |G|$*, and let V be an F-representation of G. Then every subrepresentation of V is a direct summand.*

Proof. Suppose $W \subseteq V$ is a subrepresentation. If B' is a basis of W, extend it to a basis B of B. Now define the projection of F-vector spaces $p : V \to W$ sending $p(b) = b$ if $b \in B'$ and $p(b) = 0$ if $b \notin B'$. To make this into an $F[G]$-module homomorphism, define $\pi : V \to W$ by

$$\pi(v) = \frac{1}{|G|} \sum_{g \in G} g^{-1} p(gv).$$

This is F-linear. Reindexing by $k = hg$, we get

$$\pi(hv) = \frac{1}{|G|} \sum_{g \in G} g^{-1} p(ghv) = \frac{1}{|G|} \sum_{k \in G} (hk^{-1}) p(kv) = h\pi(v),$$

so π is an $F[G]$-module homomorphism. We claim that this π splits the inclusion $W \to V$. To see this, for $w \in W$,

$$\pi(w) = \frac{1}{|G|} \sum_{g \in G} g^{-1} g w = w.$$

Since π splits $W \to V$, W is a direct summand. $\square$

Corollary 1.1. *If G is finite and $\mathrm{Char}(F) \nmid |G|$, then $F[G] \cong \prod_{i=1}^k M_{n_i}(D_i)$, where D_i is a finite dimensional division algebra with center a finite extension of F. If F is algebraically closed, then $F[G] \cong \prod_{i=1}^k M_{n_i}(F)$.*

Example 1.6. We have $\mathbb{Q}[\mathbb{Z}/p\mathbb{Z}] \cong \mathbb{Q}[x]/(x^p - 1) \cong \mathbb{Q} \times \mathbb{Q}[x]/(\Phi_p) \cong \mathbb{Q} \times \mathbb{Q}(\zeta_p)$.

Example 1.7. Since $\mathbb{C}$ is algebraically closed, we have $\mathbb{C}[\mathbb{Z}/p\mathbb{Z} \cong \mathbb{C}^p$.

Example 1.8. On the other hand, $\mathbb{F}_p[\mathbb{Z}/p\mathbb{Z}] \cong \mathbb{F}_p[x]/(x^p - 1) \cong \mathbb{F}_p[x]/(x-1)^p \cong \mathbb{F}_p[y]/(y)^p$.

Let F be finite, $\mathrm{char}(F) \nmid |G|$, and F be algebraically closed. Then we have $F[G] \cong \prod -i = 1^k M_{n_i}(F)$, and there exist k isomorphism classes of irreducible representations $V_i \cong F^{n_i}$. As $F[G]$-modules, we have $F[G] \cong V_1^{n_1} \oplus \cdots \oplus V_k^{n_k}$ as $F[G]$-modules. so $|G| = \sum_{i=1}^k n_i^2$, where $n_i = \dim_F(V_i)$.

Proposition 1.1. *Given as above, k is the number of conjugacy classes in G.*

Proof. $Z(M_{n_i}(F)) = F$, so $\dim_F(Z(F[G])) = k$. Denote C_g as the conjugacy class of $g \in G$. Let $N_g = \sum_{h \in C_g} h \in F[G]$. If $g \not\sim h$, then N_g, N_h are F-linearly independent. Also observe that $G \circlearrowright F[G]$ by conjugation: $h(\sum a_g g)h^{-1} = \sum a_g g h g^{-1}$. The invariant group under this action is $Z(F(G))$. Since N_g is fixed by this action, $N_G \in Z(F[G])$. So $k \geq$ the number of conjugacy classes.

If $z = \sum a_g g \in Z(F[G])$ and $h \in G$, then $z = \sum_g a_g hgh^{-1} = \sum_g a_{h^{-1}gh} g$, so $a_g = a_{h^{-1}gh}$ for all k. So a_g is constant on conjugacy classes. So $z \in \mathrm{span}_F(\{N_G\})$. So k is the number of conjugacy classes. $\square$

Definition 1.6. If V and W are representations of G with V semisimple and W irreducible, then the **multiplicity** of W in B is the largest n such that $W^n \subseteq B$.

Example 1.9. The group $S_3 = \{e\} \cup \{(1\,2), (1\,3), (2\,3)\} \cup \{(1\,2\,3), (1\,3\,2)\}$ has 3 conjugacy classes. We have $n_1^2 + n_2^2 + n_3^2 = |S_3| = 6$, so $n_1 = n_2 = 1$, and $n_3 = 2$ (or some permutation of this). Then $\mathbb{C}[S_3] \cong \mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C})$.

We have the trivial representation $S_3 \to \mathbb{C}^\times$ sending $\sigma \mapsto 1$ and the sign representation $\mathrm{sgn} : S_3 \to \mathbb{C}^\times$ sending $\sigma \mapsto \mathrm{sgn}(\sigma)$. What is the third representation? We know the permutation representation $S_3 \; to \, \mathrm{GL}_3(\mathbb{C})$; this has dimension 3, so it is not irreducible. On the other hand, it is not abelian, so it cannot just contain copies of our two previous 1-dimensional representations. So we can consider the subrepresentation $W = \langle e_1 - e_2, e_2 - e_3 \rangle$. Check that if $\tau = (1\,2)$, then

$$\tau(e_1 - e_2) = e_1 - e_2, \qquad \tau(e_2 - e_3) = (e_1 - e_2) + (e_2 - e_3)$$

$$\implies \rho_W(\tau) = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}.$$

If $\sigma = (1\,2\,3)$, then

$$\sigma(e_1 - e_2) = e_2 - e_3, \qquad \sigma(e_2 - e_3 = e_3 - e_1 = -(e - 1 - e_2) - (e_2 - e_3)$$

$$\implies \rho_W(\sigma) = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}.$$

Check that these matrices do not commute. So W is the 2-dimensional irreducible representation we are looking for. These are all $\mathbb{Q}$-representations, as well, so $\mathbb{Q}[S_3] \cong \mathbb{Q} \times \mathbb{Q} \times M_2(\mathbb{Q})$.